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Positon and negaton solutions of the mKdV equation with self-consistent sources

Hongxia Wu[1,2], **Yunbo Zeng**[3] **and Tianyou Fan**[2]

[1] Department of Mathematics, School of Science, Jimei University, Xiamen 361021, People's Republic of China
[2] Department of Mathematics, School of Science, Beijing Institute of Technology, Beijing 100081, People's Republic of China
[3] Department of Mathematical Sciences, Tsinghua University, Beijing 100084, People's Republic of China

E-mail: wuhongxia796@163.com

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Abstract

By using the generalized binary Darboux transformation with arbitrary functions at time t for the negative modified KdV equation with self-consistent sources (mKdV$^-$ESCSs) which offers a non-auto-Bäcklund transformation between two mKdV$^-$ESCSs with different degrees of sources, some new solutions for the mKdV$^-$ESCSs such as singular multisoliton, multipositon, multinegaton, multisoliton–positon, multisoliton–negaton and multipositon–negaton solutions are found by taking the special initial solution for auxiliary linear problems and the special functions of t-time. At the same time, the properties of these solutions are analyzed in detail.

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1. Introduction

It is well known that the soliton equations with self-consistent sources (SESCSs) have important physical applications. For example, the KdV equation with self-consistent sources describes the interaction of long–short capillary-gravity waves [1] and the nonlinear Schrödinger equation with self-consistent sources describes the soliton propagation in a medium with both resonant and nonresonant nonlinearities [2, 3] as well as the nonlinear interaction of high-frequency electrostatic waves with ion acoustic waves in plasma [4]. Therefore SESCSs have attracted some attention [5–13]. In recent years, SESCSs were studied in the framework of the high-order constrained flows of soliton equations [14–16], namely the high-order constrained flows of soliton equations are considered as the stationary equations of the SESCSs. These SESCSs can be solved by the inverse scattering method [4–6, 12], Darboux transformation [17–22] and Hirota method and Wronskian technique [23–27].

The Darboux transformation (DT) is a powerful tool for solving soliton equations [28]. However, for soliton equations with sources the normal DT cannot be used to construct the nontrivial solution from the trivial solution seed. Zeng and others [17–19] proposed a generalized binary Darboux transformation with an arbitrary function at time t for SESCSs, which offers a non-auto-Bäcklund transformation between two SESCSs with different degrees of sources. This kind of DT enables us to obtain soliton, negaton and positon solutions for SESCSs.

The study of negaton and positon solutions has been made for the KdV equation [29]. It was pointed out that negaton was a singular reduced two-soliton while the positon solution was a long-range, slowly decreasing and oscillating singular soliton-like solution. It was also shown that positon was absolutely transparent for soliton and negaton, i.e., soliton and negaton gain no phase shifts when colliding with positon, and that two positons were totally insensitive to the mutual collision, even without additional phase shifts. Some positons and negatons are obtained for many soliton equations by using the Darboux transformation [18, 20, 21, 29, 31–34]. By Hirota's bilinear method, rational solutions, solitons, negatons and positons are recovered for the KdV equation [30], the KdV equation with sources [35], the Schrödinger source equation [36], etc from their Wronskian and so-called generalized Wronskian solutions. Moreover, a more general class of exact solutions to the KdV equation, called complexiton solutions, is furnished by the Wronskian or Casorati formulation for the KdV equation [37, 38] and Toda lattice [39]. However, up to now, positon and negaton solutions for the mKdV equation with self-consistent source (mKdVESCSs) have not been investigated though its N-soliton solution has been obtained by the integral-type Darboux transformations [22] and Hirota method and Wronskian technique [24].

In this paper, by reducing the generalized binary Darboux transformation with an arbitrary function at time t for the AKNS equation with self-consistent sources (AKNSESCSs) presented in [20], we obtain the generalized binary Darboux transformation with arbitrary functions at time t for the negative modified KdV equation with self-consistent sources (mKdV$^-$ESCSs) which offers a non-auto-Bäcklund transformation between two mKdV$^-$ESCSs with different degrees of sources. Some new solutions for the mKdV$^-$ESCSs such as singular multisoliton, multipositon, multinegaton, multisoliton–positon, multisoliton–negaton and multipositon–negaton solutions are constructed by taking the special initial solution for auxiliary linear problems and the special functions of time t, which greatly enriches the solution structure of the mKdVESCSs. In addition, the properties of these solutions are analyzed in detail.

2. Generalized binary Darboux transformation for the AKNSESCSs

First, we briefly review the multi-times repeated generalized binary Darboux transformation with an arbitrary function of t for the AKNS equation with self-consistent sources.

The third equation in AKNS hierarchy with self-consistent sources (ANKSESCSs) is defined as [12, 13]

$$q_t = 6qq_xr - q_{xxx} + \sum_{j=1}^{n} \left(\varphi_j^{(1)}\right)^2, \tag{2.1a}$$

$$r_t = 6qrr_x - r_{xxx} + \sum_{j=1}^{n} \left(\varphi_j^{(2)}\right)^2 \tag{2.1b}$$

$$\varphi_{j,x} = U(\lambda_j, q, r)\varphi_j, \qquad j = 1, \ldots, n, \tag{2.1c}$$

where

$$U(\lambda, q, r) = \begin{pmatrix} -\lambda & q \\ r & \lambda \end{pmatrix}$$

λ_j, $j = 1, \ldots, n$, are n distinct complex constants, $\varphi_j = \left(\varphi_j^{(1)}, \varphi_j^{(2)}\right)^T$.

The corresponding Lax pair for equations (2.1) is given by [12, 13]

$$\psi_x = U(\lambda, q, r)\psi, \tag{2.2a}$$

$$\psi_t = R^{(n)}(\lambda, q, r)\psi, \qquad R^{(n)}(\lambda, q, r) = V(\lambda, q, r) + \sum_{j=1}^{n} \frac{H(\varphi_j)}{\lambda - \lambda_j}, \tag{2.2b}$$

where

$$V(\lambda, q, r) = \begin{pmatrix} 4\lambda^3 - 2\lambda qr - qr_x + rq_x & -4\lambda^2 q + 2\lambda q_x - q_{xx} + 2q^2 r \\ -4\lambda^2 r - 2\lambda r_x - r_{xx} + 2r^2 q & -4\lambda^3 + 2\lambda qr + qr_x - rq_x \end{pmatrix} \tag{2.2c}$$

$$H(\varphi_j) = \frac{1}{2} \begin{pmatrix} -\varphi_j^{(1)}\varphi_j^{(2)} & \left(\varphi_j^{(1)}\right)^2 \\ -\left(\varphi_j^{(2)}\right)^2 & \varphi_j^{(1)}\varphi_j^{(2)} \end{pmatrix}. \tag{2.2d}$$

In order to obtain the N-times repeated Darboux transformation with an arbitrary function of t, some symmetric forms are defined.

Let c_j be a series of scalar and $f_j = \binom{f_j^{(1)}}{f_j^{(2)}}$ are the solutions of (2.2) with $\lambda = \lambda_j$, $j = 1, \ldots, N$, u be a scalar, $h = \binom{h^{(1)}}{h^{(2)}}$, then W_0, $W_1^{(i)}$, $W_2^{(i)}$, $i = 1, 2$ and W_1 are defined as follows:

$$\Delta = W_0(\{c_1, f_1\}, \ldots, \{c_N, f_N\}) = \det A$$

$$W_1^{(i)}(\{c_1, f_1\}, \ldots, \{c_N, f_N\}; h) = \det \begin{pmatrix} A & b \\ \alpha^{(i)} & h^{(i)} \end{pmatrix}, \qquad i = 1, 2$$

$$W_2^{(i)}(\{c_1, f_1\}, \ldots, \{c_N, f_N\}; u) = \det \begin{pmatrix} A & (\alpha^{(i)})^T \\ \alpha^{(i)} & u \end{pmatrix}, \qquad i = 1, 2,$$

$$W_1(\{c_1, f_1\}, \ldots, \{c_N, f_N\}; h) = \begin{pmatrix} W_1^{(1)}(\{c_1, f_1\}, \ldots, \{c_N, f_N\}; h) \\ W_1^{(2)}(\{c_1, f_1\}, \ldots, \{c_N, f_N\}; h) \end{pmatrix}$$

where

$$A = (\delta_{ij} c_i + \sigma(f_i, f_j))_{N \times N}, \qquad b = (\sigma(f_1, h), \ldots, \sigma(f_N, h))^T, \qquad \alpha^{(i)} = \left(f_1^{(i)}, \ldots, f_N^{(i)}\right)$$

$$\sigma(f_i, f_j) := -\frac{W(f_i, f_j)}{2(\lambda_i - \lambda_j)}, \qquad \sigma(f_j, f_j) := \frac{1}{2} W(f_j, \partial_{\lambda_j} f_j),$$

$$W(f_i, f_j) = f_i^{(1)} f_j^{(2)} - f_i^{(2)} f_j^{(1)}.$$

The N-times repeated generalized binary Darboux transformation with an arbitrary function of t for (2.1) is described by the following proposition [20].

Proposition 2.1. *Let f_j be the solutions of (2.2) with $\lambda = \lambda_{n+j}$, and let $c_j(t)$ be an arbitrary function of $t, j = 1, \ldots, N$, then the N-times repeated generalized binary Darboux transformation for (2.1) is given by*

$$\psi[N] = \frac{1}{\Delta} W_1(\{c_1, f_1\}, \ldots, \{c_N, f_N\}; \psi) \tag{2.3a}$$

$$q[N] = \frac{1}{\Delta} W_2^{(1)}(\{c_1, f_1\}, \ldots, \{c_N, f_N\}; q) \tag{2.3b}$$

$$r[N] = \frac{1}{\Delta} W_2^{(2)}(\{c_1, f_1\}, \ldots, \{c_N, f_N\}; r) \tag{2.3c}$$

$$\varphi_j[N] = \frac{1}{\Delta} W_1(\{c_1, f_1\}, \ldots, \{c_N, f_N\}; \varphi_j), \qquad j = 1, \ldots, n \tag{2.3d}$$

$$\varphi_{n+m}[N] = \frac{\sqrt{\dot{c}_m(t)}}{c_m(t)\Delta} W_1(\{c_1, f_1\}, \ldots, \{c_N, f_N\}; f_m), \qquad m = 1, \ldots, N, \tag{2.3e}$$

namely the new variables $\psi[N], q[N], r[N], \varphi_1[N], \ldots, \varphi_{n+N}[N]$ satisfy system (2.2) with n replaced by $n + N$, and $(q[N], r[N], \varphi_1[N], \ldots, \varphi_{n+N}[N])$ is a solution of (2.1) with n replaced by $n + N$.

Obviously, the N-times repeated generalized binary Darboux transformation (2.3) provides a non-auto-Bäcklund transformation between the two AKNSESCSs of degrees n and $n + N$.

3. Generalized binary Darboux transformations for the mKdV$^-$ESCSs

The ordinary AKNS equation

$$q_t = 6qq_x r - q_{xxx}, \qquad r_t = 6qrr_x - r_{xxx} \tag{3.1}$$

is reduced to the mKdV equations by setting $r = \pm q$,

$$q_t + 6\varepsilon q^2 q_x + q_{xxx} = 0, \qquad \varepsilon = \pm 1. \tag{3.2}$$

Equation (3.2) with $\varepsilon = 1$ ($\varepsilon = -1$) is denoted by the mKdV$^+$equation (mKdV$^-$equation).

Similarly, we can reduce the AKNSESCSs (2.1) to the mKdV$^-$ESCSs, but the reductions are more complicated since the sources need to be reduced consistently as well.

In order to obtain the mKdV$^-$ESCSs from the reduction of the AKNSESCSs, we have to consider the following AKNSESCSs:

$$q_t = 6qq_x r - q_{xxx} + \sum_{j=1}^{n} \left[\left(\varphi_j^{(1)}\right)^2 + \left(\omega_j^{(1)}\right)^2 \right], \tag{3.3a}$$

$$r_t = 6qrr_x - r_{xxx} + \sum_{j=1}^{n} \left[\left(\varphi_j^{(2)}\right)^2 + \left(\omega_j^{(2)}\right)^2 \right] \tag{3.3b}$$

$$\varphi_{j,x} = U(\lambda_j, q, r)\varphi_j, \qquad j = 1, \ldots, n \tag{3.3c}$$

$$\omega_{j,x} = U(\widetilde{\lambda}_j, q, r)\omega_j, \qquad j = 1, \ldots, n, \tag{3.3d}$$

where λ_j, $\widetilde{\lambda}_j$ $(j = 1, \ldots, n)$ are $2n$ distinct complex constants, $\varphi_j = \left(\varphi_j^{(1)}, \varphi_j^{(2)}\right)^T$, $\omega_j = \left(\omega_j^{(1)}, \omega_j^{(2)}\right)^T$. The corresponding Lax pair is

$$\psi_x = U(\lambda, q, r)\psi, \tag{3.4a}$$

$$\psi_t = V(\lambda, q, r)\psi + \sum_{j=1}^{n} \left[\frac{H(\varphi_j)}{\lambda - \lambda_j} + \frac{H(\omega_j)}{\lambda - \widetilde{\lambda}_j} \right] \psi. \tag{3.4b}$$

First, we define a linear map

$$S : \begin{pmatrix} \varphi_j^{(1)} \\ \varphi_j^{(2)} \end{pmatrix} \rightarrow \begin{pmatrix} \varphi_j^{(2)} \\ \varphi_j^{(1)} \end{pmatrix}. \tag{3.5}$$

In order to obtain the consistent reduction of source for $r = q$, we have to take $\widetilde{\lambda}_j = -\lambda_j$, $\omega_j = S\varphi_j$, $j = 1, \ldots, n$, then equations (3.3) are reduced to the mKdV$^-$ESCSs

$$\begin{aligned} q_t &= 6q^2 q_x - q_{xxx} + \sum_{j=1}^{n} \left[\left(\varphi_j^{(1)}\right)^2 + \left(\varphi_j^{(2)}\right)^2 \right] \\ \varphi_{j,x} &= U(\lambda_j, q, q)\varphi_j, \qquad j = 1, \ldots, n. \end{aligned} \tag{3.6}$$

And system (3.4) is reduced to the Lax pair for the mKdV$^-$ESCSs

$$\begin{aligned} \psi_x &= U(\lambda, q, q)\psi \\ \psi_t &= V(\lambda, q, q)\psi + \sum_{j=1}^{n} \left[\frac{H(\varphi_j)}{\lambda - \lambda_j} + \frac{H(S\varphi_j)}{\lambda + \lambda_j} \right] \psi. \end{aligned} \tag{3.7}$$

We now reduce the Darboux transformation for the AKNSESCSs to that for the mKdV$^-$ESCSs.

Proposition 3.1. *Let $(q, \varphi_1, \ldots, \varphi_n)$ be a solution of equations (3.6), $c(t)$ be an arbitrary t-dependent function and f be a solution of (3.7) with $\lambda = \lambda_{n+1}$. The generalized binary Darboux transformation with an arbitrary function of t for the mKdV$^-$ESCSs is given by*

$$\Delta = W_0(\{c, f\}, \{c, Sf\}) \tag{3.8a}$$

$$\overline{\psi} = \Delta^{-1} W_1(\{c, f\}, \{c, Sf\}; \psi) \tag{3.8b}$$

$$\overline{q} = q + \Delta^{-1} W_2^{(1)}(\{c, f\}, \{c, Sf\}; 0) \tag{3.8c}$$

$$\overline{\varphi_j} = \Delta^{-1} W_1(\{c, f\}, \{c, Sf\}; \varphi_j), \qquad j = 1, \ldots, n \tag{3.8d}$$

$$\overline{\varphi}_{n+1} = \frac{\sqrt{\dot{c}(t)}}{c(t)\Delta} W_1(\{c, f\}, \{c, Sf\}; f), \tag{3.8e}$$

namely $\overline{\psi}, \overline{q}, \overline{\varphi}_1, \ldots, \overline{\varphi}_{n+1}$ satisfy equation (3.7) with n replaced by $n+1$, and $(\overline{q}, \overline{\varphi}_1, \ldots, \overline{\varphi}_{n+1})$ is a solution of equation (3.6) with n replaced by $n + 1$.

The multi-times repeated generalized binary Darboux transformation with an arbitrary function of t for the mKdV$^-$ESCSs is given by the following proposition.

Proposition 3.2. *Let $(q, \varphi_1, \ldots, \varphi_n)$ be a solution of equations (3.6), $c_j(t)$ be an arbitrary function of t and f_j be a solution of (3.7) with $\lambda = \lambda_{n+j}$, $j = 1, \ldots, N$. Then the multi-times repeated generalized binary Darboux transformation with an arbitrary function of t for the mKdV$^-$ESCSs is given by*

$$\Delta = W_0(\{c_1, f_1\}, \{c_1, Sf_1\}, \ldots, \{c_N, f_N\}, \{c_N, Sf_N\}) \tag{3.9a}$$

$$\psi[N] = \Delta^{-1} W_1(\{c_1, f_1\}, \{c_1, Sf_1\}, \ldots, \{c_N, f_N\}, \{c_N, Sf_N\}; \psi) \tag{3.9b}$$

$$q[N] = q + \Delta^{-1} W_2^{(1)}(\{c_1, f_1\}, \{c_1, Sf_1\}, \ldots, \{c_N, f_N\}, \{c_N, Sf_N\}; 0) \tag{3.9c}$$

$$\varphi_j[N] = \Delta^{-1} W_1(\{c_1, f_1\}, \{c_1, Sf_1\}, \ldots, \{c_N, f_N\}, \{c_N, Sf_N\}; \varphi_j), \qquad j = 1, \ldots, n \tag{3.9d}$$

$$\varphi_{n+m}[N] = \frac{\sqrt{\dot{c}_m(t)}}{c_m(t)\Delta} W_1(\{c_1, f_1\}, \{c_1, Sf_1\}, \ldots, \{c_N, f_N\}, \{c_N, Sf_N\}; f_m), \quad m = 1, \ldots, N, \tag{3.9e}$$

namely $\psi[N], q[N], \varphi_1[N], \ldots, \varphi_{n+N}[N]$ *satisfy equation (3.7) with nreplaced by* $n + N$*, and* $(q[N], \varphi_1[N], \ldots, \varphi_{n+N}[N])$ *is a solution of equation (3.6) with n replaced by* $n + N$.

4. Solutions of the mKdV$^-$ESCSs

This section aims at applying the Darboux transformations (3.8) and (3.9) to obtain some new solutions, such as singular soliton, positon and negaton solutions for the mKdV$^-$ESCSs. At the same time, the properties of these new solutions are analyzed in detail.

4.1. Singular soliton solutions

We now use proposition 3.1 to construct the singular one-soliton solution for the mKdV$^-$ESCSs.

We take a solution of equation (3.7) with $q = 0, n = 0$ and let $\lambda_1 \in R$ as follows:

$$f_1 = \begin{pmatrix} \exp(-\theta_1) \\ 0 \end{pmatrix}, \qquad Sf_1 = \begin{pmatrix} 0 \\ \exp(-\theta_1) \end{pmatrix} \tag{4.1a}$$

$$c_1(t) = -\frac{\exp(-\lambda_1\alpha_1 t)}{4\lambda_1}, \qquad \alpha_1 \geqslant 0, \tag{4.1b}$$

$$\text{where} \quad \theta_1 = \lambda_1\left(x - 4\lambda_1^2 t + x_1\right), \qquad x_1 \text{ is an analytic function of } \lambda_1. \tag{4.1c}$$

From (3.8*c*)–(3.8*e*) with $q = 0$, $n = 0$, we obtain singular one-soliton solution for the mKdV$^-$ESCSs with $n = 1$:

$$q = \frac{2\lambda_1}{\sinh(2\overline{\theta_1})} \tag{4.2a}$$

$$\varphi_1^{(1)} = -\frac{\lambda_1\sqrt{\alpha_1}\exp(\overline{\theta_1})}{\sinh(2\overline{\theta_1})}, \qquad \varphi_1^{(2)} = \frac{\lambda_1\sqrt{\alpha_1}\exp(-\overline{\theta_1})}{\sinh(2\overline{\theta_1})}, \tag{4.2b}$$

where $\overline{\theta_1} = \lambda_1\left(x - 4\lambda_1^2 t - \frac{\alpha_1}{2} t + x_1\right)$.

Obviously, the singularity of the one-soliton solution (4.2) is determined by $\sinh(2\overline{\theta_1}) = 0$.

Similarly, in order to find a two-soliton solution, we take the solution of (3.7) with $q = 0, n = 0$ as follows:

$$f_j = \begin{pmatrix} \exp(-\theta_j) \\ 0 \end{pmatrix}, \qquad Sf_j = \begin{pmatrix} 0 \\ \exp(-\theta_j) \end{pmatrix}, \qquad \lambda_j \in R, \qquad j = 1, 2,$$

$$\theta_j = \lambda_j\left(x - 4\lambda_j^2 t + x_j\right), \qquad c_j(t) = -\frac{\exp(-\lambda_j\alpha_j t)}{4\lambda_j}, \qquad \alpha_j \geqslant 0,$$

then we have

$$\Delta = c_1^2 c_2^2 - \frac{c_2^2 \exp(-4\theta_1)}{16\lambda_1^2} - \frac{c_1^2 \exp(-4\theta_2)}{16\lambda_2^2} + \frac{(\lambda_1-\lambda_2)^4 \exp(-4\theta_1-4\theta_2)}{256\lambda_1^2\lambda_2^2(\lambda_1+\lambda_2)^4} - \frac{c_1 c_2 \exp(-2\theta_1-2\theta_2)}{2(\lambda_1+\lambda_2)^2} \tag{4.3a}$$

$$W_2^{(1)}(\{c_1, f_1\}, \{c_1, Sf_1\}, \{c_2, f_2\}, \{c_2, Sf_2\}; 0) = c_1^2 c_2 \exp(-2\theta_2) - c_1 c_2^2 \exp(-2\theta_1) + \frac{c_2(\lambda_1-\lambda_2)^2}{16\lambda_1^2(\lambda_1+\lambda_2)^2}\exp(-4\theta_1-2\theta_2) + \frac{c_1(\lambda_1-\lambda_2)^2}{16\lambda_2^2(\lambda_1+\lambda_2)^2}\exp(-2\theta_1-4\theta_2) \tag{4.3b}$$

$$W_1^{(1)}\left(\{c_1, f_1\}, \{c_1, Sf_1\}, \{c_2, f_2\}, \{c_2, Sf_2\}; f_i^{(1)}\right) = c_1^2 c_2^2 \exp(-\theta_i) + \frac{c_1 c_2(\lambda_i-\lambda_{i+1})\exp(-3\theta_i-2\theta_{i+1})}{8\lambda_i(\lambda_1+\lambda_2)^2} + \frac{c_i^2(\lambda_{i+1}-\lambda_i)}{16\lambda_{i+1}^2(\lambda_1+\lambda_2)^2}\exp(-\theta_i-4\theta_{i+1}) \tag{4.3c}$$

$$W_1^{(2)}\left(\{c_1, f_1\}, \{c_1, Sf_1\}, \{c_2, f_2\}, \{c_2, Sf_2\}; f_2^{(i)}\right) = \frac{c_i c_{i+1}^2}{4\lambda_i}\exp(-3\theta_i) + \frac{c_i(\lambda_i-\lambda_{i+1})^3\exp(-3\theta_i-4\theta_{i+1})}{64\lambda_i\lambda_{i+1}^2(\lambda_1+\lambda_2)^3} + \frac{c_i^2 c_{i+1}\exp(-\theta_i-2\theta_{i+1})}{2(\lambda_1+\lambda_2)} \tag{4.3d}$$

where $i = 1, 2$ and $i+1 = \begin{cases} 2, & i+1=2 \\ 1, & i+1>2 \end{cases}$.

Then (3.9c)–(3.9e) with $q = 0$, $n = 0$ and $N = 2$ give rise to the singular two-soliton solution, $q[2]$, $\varphi_1[2]$, $\varphi_2[2]$ for the mKdV$^-$ESCSs (3.6) with $n = 2$.

In the domain where $\theta_2 = \lambda_2\left(x - 4\lambda_2^2 t + x_2\right)$ is fixed and $t \to \pm\infty$, the asymptotic solution is

$$q[2] \sim \frac{2\lambda_2}{\sinh(2\overline{\theta_2})}, \qquad \varphi_2[2] \sim \begin{pmatrix} -\frac{\lambda_2\sqrt{\alpha_2}\exp(\overline{\theta_2})}{\sinh(2\overline{\theta_2})} \\ \frac{\lambda_2\sqrt{\alpha_2}\exp(-\overline{\theta_2})}{\sinh(2\overline{\theta_2})} \end{pmatrix}, \qquad \varphi_1[2] \sim \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \qquad t \to -\infty \tag{4.4a}$$

$$q[2] \sim \frac{2\lambda_2}{\sinh 2\left(\overline{\theta_2} + \frac{1}{2}\varepsilon_0\right)}, \qquad \varphi_2[2] \sim \begin{pmatrix} -\frac{\lambda_2\sqrt{\alpha_2}\exp\left(\overline{\theta_2}+\frac{1}{2}\varepsilon_0\right)}{\sinh 2\left(\overline{\theta_2}+\frac{1}{2}\varepsilon_0\right)} \\ \frac{\lambda_2\sqrt{\alpha_2}\exp\left[-\left(\overline{\theta_2}+\frac{1}{2}\varepsilon_0\right)\right]}{\sinh 2\left(\overline{\theta_2}+\frac{1}{2}\varepsilon_0\right)} \end{pmatrix}, \qquad \varphi_1[2] \sim \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \qquad t \to \infty, \tag{4.4b}$$

where $\varepsilon_0 = \ln\frac{(\lambda_1-\lambda_2)^2}{(\lambda_1+\lambda_2)^2}$, $\overline{\theta_2} = \lambda_2\left(x - 4\lambda_2^2 t - \frac{\alpha_2}{2}t + x_2\right)$.

When θ_1 is fixed and $t \to \pm\infty$, we have a similar result for the asymptotic solution. These estimates show that, in the indicated domain, the shape of the singular two-soliton emerging out of the interaction is completely unchanged, except for the phase shift $\pm\frac{1}{k_{1,2}}\ln\frac{k_1-k_2}{k_1+k_2}$.

The singular$N$-soliton solution of equation (3.6) with $n = N$ and real λ_j, $j = 1, \ldots, N$, is given by (3.9c)–(3.9e) with $q = 0$, $n = 0$ and

$$f_j = \begin{pmatrix} \exp(-\theta_j) \\ 0 \end{pmatrix}, \qquad Sf_j = \begin{pmatrix} 0 \\ \exp(-\theta_j) \end{pmatrix}, \qquad j = 1, \ldots, N$$

$$c_j(t) = -\frac{\exp(-\lambda_j\alpha_j t)}{4\lambda_j}, \qquad \alpha_j \geqslant 0, \qquad \theta_j = \lambda_j\left(x - 4\lambda_j^2 t + x_j\right).$$

4.2. Negaton solutions

Hereafter, we always take $c_j(t) = a_j t + b_j$, where $a_j \neq 0$ and b_j are real constants.

4.2.1. One-negaton solution. In contrast with (4.1), for obtaining the one-negaton solution we take the solution of system (3.7) with $q = 0, n = 0$ and $\lambda = \lambda_1$, $\mathrm{Im}\,\lambda_1 = 0$ as

$$f_1 = \begin{pmatrix} \exp(-\theta_1) \\ \exp\theta_1 \end{pmatrix}, \qquad Sf_1 = \begin{pmatrix} \exp(\theta_1) \\ \exp(-\theta_1) \end{pmatrix} \qquad (4.5)$$
$$c_1(t) = a_1 t + b_1, \quad a_1 \neq 0 \quad \text{and} \quad b_1 \text{ are real constants}$$

where θ_1 is defined by (4.1*c*).

According to (3.8*c*)–(3.8*e*) with $q = 0, n = 0$ and $N = 1$, we obtain the one-negaton solution for the mKdV$^-$ESCSs (3.6) with $n = 1$:

$$q = \frac{4\lambda_1(2\lambda_1 r_1 \cosh 2\theta_1 - \sinh 2\theta_1)}{\sinh^2(2\theta_1) - 4\lambda_1^2 r_1^2} \qquad (4.6a)$$

$$\varphi_1^{(1)} = \frac{2\sqrt{a_1}\lambda_1(\exp(\theta_1)\sinh(2\theta_1) - 2\lambda_1 r_1 \exp(-\theta_1))}{\sinh^2(2\theta_1) - 4\lambda_1^2 r_1^2} \qquad (4.6b)$$

$$\varphi_1^{(2)} = \frac{2\sqrt{a_1}\lambda_1(\exp(-\theta_1)\sinh(2\theta_1) - 2\lambda_1 r_1 \exp(\theta_1))}{\sinh^2(2\theta_1) - 4\lambda_1^2 r_1^2}, \qquad (4.6c)$$

where $r_1 = x + (x_1 + \lambda_1 \partial_{\lambda_1} x_1) - \left(12\lambda_1^2 - a_1\right)t + b_1$.

As a function of x, q, $\varphi_1^{(1)}$ and $\varphi_1^{(2)}$ have two one-order poles which locate at the points $x = x_p(t)$ determined by the equations $\sinh(2\theta_1) = 2\lambda_1 r_1$ and $\sinh(2\theta_1) = -2\lambda_1 r_1$, respectively. The shape and the motion of $q(x, t)$ is the same as that described in [31].

4.2.2. Two-negaton solution. The two-negaton solution of equation (3.6) with $n = 2$ and real λ_j, $j = 1, 2$, is given by (3.9*c*)–(3.9*e*) by taking $q = 0, n = 0, N = 2$ and

$$f_j = \begin{pmatrix} \exp(-\theta_j) \\ \exp\theta_j \end{pmatrix}, \qquad \theta_j = \lambda_j\left(x - 4\lambda_j^2 t + x_j\right), \qquad c_j(t) = a_j t + b_j, \qquad j = 1, 2.$$

From (3.9*c*)–(3.9*e*) with $q = 0, n = 0, N = 2$, we easily obtain the asymptotic behavior of the two-negaton solution for the mKdV$^-$ESCSs (3.6) with $n = 2$.

In the domain where $\theta_2 = \lambda_2\left(x - 4\lambda_2^2 t + x_2\right)$ is fixed and $t \to \pm\infty$, the asymptotic solution is

$$q[2] \sim \frac{4\lambda_2[2\lambda_2(r_2 - r_0)\cosh 2(\theta_2 + \theta_0) - \sinh 2(\theta_2 + \theta_0)]}{\sinh^2 2(\theta_2 + \theta_0) - 4\lambda_2^2(r_2 - r_0)^2}$$

$$\varphi_2[2] \sim \begin{pmatrix} \dfrac{2\sqrt{a_2}\lambda_2(\exp(\theta_2 + \theta_0)\sinh 2(\theta_2 + \theta_0) - 2\lambda_2(r_2 - r_0)\exp[-(\theta_2 + \theta_0)])}{\sinh^2 2(\theta_2 + \theta_0) - 4\lambda_2^2(r_2 - r_0)^2} \\ \dfrac{2\sqrt{a_2}\lambda_2(\exp[-(\theta_2 + \theta_0)]\sinh 2(\theta_2 + \theta_0) - 2\lambda_2(r_2 - r_0)\exp(\theta_2 + \theta_0))}{\sinh^2 2(\theta_2 + \theta_0) - 4\lambda_2^2(r_2 - r_0)^2} \end{pmatrix},$$

$$\varphi_1[2] \sim \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$t \to -\infty$$

$$q[2] \sim \frac{4\lambda_2[2\lambda_2(r_2+r_0)\cosh 2(\theta_2-\theta_0) - \sinh 2(\theta_2-\theta_0)]}{\sinh^2 2(\theta_2-\theta_0) - 4\lambda_2^2(r_2+r_0)^2}$$

$$\varphi_2[2] \sim \begin{pmatrix} \dfrac{2\sqrt{a_2}\lambda_2(\exp(\theta_2-\theta_0)\sinh 2(\theta_2-\theta_0) - 2\lambda_2(r_2+r_0)\exp[-(\theta_2-\theta_0)])}{\sinh^2 2(\theta_2-\theta_0) - 4\lambda_2^2(r_2+r_0)^2} \\ \dfrac{2\sqrt{a_2}\lambda_2(\exp[-(\theta_2-\theta_0)]\sinh 2(\theta_2-\theta_0) - 2\lambda_2(r_2+r_0)\exp(\theta_2-\theta_0))}{\sinh^2 2(\theta_2-\theta_0) - 4\lambda_2^2(r_2+r_0)^2} \end{pmatrix},$$

$$\varphi_1[2] \sim \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$t \to +\infty,$$

where $r_0 = \frac{2\lambda_1}{\lambda_1^2-\lambda_2^2}$, $\theta_0 = \ln \frac{\lambda_1-\lambda_2}{\lambda_1+\lambda_2}$, $r_2 = x + (x_2 + \lambda_2 \partial_{\lambda_2} x_2) - \left(12\lambda_2^2 - a_2\right)t + b_2$.

When $\theta_1 = \lambda_1\left(x - 4\lambda_1^2 t + x_1\right)$ is fixed and $t \to \pm\infty$, we have a similar result for the asymptotic solution. These estimates show that, in the indicated domain, the shape of the two-negaton emerging out of the interaction is completely unchanged, except for phase shifts. We can find two different phase shifts for negaton, one $-2r_0$ in the linear function, another one $2\theta_0$ in the hyperbolic sine and exponent functions.

4.2.3. Multinegaton solutions. The N-negaton solution of the mKdV$^-$ESCSs (3.6) with $n = N$ and real λ_j, $j = 1, \ldots, N$, is given by (3.9*c*) and (3.9*e*) by taking $q = 0$, $n = 0$ and

$$f_j = \begin{pmatrix} \exp(-\theta_j) \\ \exp\theta_j \end{pmatrix}, \qquad \theta_j = \lambda_j\left(x - 4\lambda_j^2 t + x_j\right), \qquad c_j(t) = a_j t + b_j, \quad j = 1, \ldots, N.$$

4.3. Positon solutions

4.3.1. One-positon solution. Let f_1 be a solution of equation (3.7) with $q = 0$, $n = 0$ and $\lambda = \lambda_1 = \mathrm{i}k_1$, k_1 is real:

$$f_1 = \begin{pmatrix} \exp(-\mathrm{i}\theta_1) \\ \exp(\mathrm{i}\theta_1) \end{pmatrix}, \qquad Sf_1 = \begin{pmatrix} \exp(\mathrm{i}\theta_1) \\ \exp(-\mathrm{i}\theta_1) \end{pmatrix}, \tag{4.7}$$

where $\theta_1 = k_1\left(x + 4k_1^2 t + x_1\right)$, x_1 is an analytic function of k_1. Then the binary Darboux transformation with an arbitrary function of t for the mKdV$^-$ESCSs (3.6) gives

$$q = -\frac{4k_1(\sin 2\theta_1 - 2k_1 r_1 \cos 2\theta_1)}{\sin^2(2\theta_1) - 4k_1^2 r_1^2} \tag{4.8a}$$

$$\varphi_1^{(1)} = \frac{2\sqrt{a_1}k_1(2k_1 r_1 \exp(-\mathrm{i}\theta_1) - \exp(\mathrm{i}\theta_1)\sin(2\theta_1))}{\sin^2(2\theta_1) - 4k_1^2 r_1^2} \tag{4.8b}$$

$$\varphi_1^{(2)} = \frac{2\sqrt{a_1}k_1(2k_1 r_1 \exp(\mathrm{i}\theta_1) - \exp(-\mathrm{i}\theta_1)\sin(2\theta_1))}{\sin^2(2\theta_1) - 4k_1^2 r_1^2}, \tag{4.8c}$$

with $r_1 = x + \tilde{x}_1 + \left(12k_1^2 + a_1\right)t + b_1$, $\tilde{x}_1 = x_1 + k_1 \partial_{k_1} x_1$, which gives the one-positon solution of the mKdV$^-$ESCSs with $n = 1$, $\lambda_1 = \mathrm{i}k_1$, $\mathrm{Im}\, k_1 = 0$ corresponding to the one-positon solution of the mKdV equation in [31, 32].

Based on the formulae (4.8), we can analyze the basic features of the one-positon solution of equation (3.6) in the same way as in [31, 32]. We can conclude that for the one-positon solution of (3.6) with $n = 1$, $q(x, t)$ has the same shape, the same asymptotic behavior when $t \to \pm\infty$ as the one-positon solution of the mKdV equation, i.e. long-range analogs of solitons of the mKdV$^-$ESCSs and slowly decreasing, oscillating solutions. We will show in the following that the one-positon potential is superreflectionless, namely the corresponding reflection coefficient is zero and the transmission coefficient is unity.

From proposition 3.1, we obtain

$$W_1^{(1)}(\{c_1, f_1\}, \{c_1, S_1 f_1\}; \psi) = \exp(-\mathrm{i}\theta) r_1^2 - \frac{\exp(-\mathrm{i}\theta_1) r_1 \sin(\theta - \theta_1)}{k - k_1} - \frac{\exp(-\mathrm{i}\theta)\sin(2\theta_1)}{4k_1^2} + \frac{\exp(\mathrm{i}\theta_1)\sin(2\theta_1)\sin(\theta - \theta_1)}{2k_1(k - k_1)} - \frac{\exp(\mathrm{i}\theta_1) r_1 \sin(\theta + \theta_1)}{k + k_1} + \frac{\exp(-\mathrm{i}\theta_1)\sin(2\theta_1)\sin(\theta + \theta_1)}{2k_1(k + k_1)}$$

$$W_1^{(2)}(\{c_1, f_1\}, \{c_1, S_1 f_1\}; \psi) = \exp(\mathrm{i}\theta) r_1^2 - \frac{\exp(\mathrm{i}\theta_1) r_1 \sin(\theta - \theta_1)}{k - k_1} - \frac{\exp(\mathrm{i}\theta)\sin(2\theta_1)}{4k_1^2} + \frac{\exp(-\mathrm{i}\theta_1)\sin(2\theta_1)\sin(\theta - \theta_1)}{2k_1(k - k_1)} - \frac{\exp(-\mathrm{i}\theta_1) r_1 \sin(\theta + \theta_1)}{k + k_1} + \frac{\exp(\mathrm{i}\theta_1)\sin(2\theta_1)\sin(\theta + \theta_1)}{2k_1(k + k_1)}$$

$$\Delta = r_1^2 - \frac{\sin^2(2\theta_1)}{4k_1^2}$$

$$\overline{\psi}^{(1)} = \frac{W_1^{(1)}(\{c_1, f_1\}, \{c_1, S_1 f_1\}; \psi)}{\Delta} \sim \exp[-\mathrm{i}k(x + 4k^2 t + \widetilde{x})], \qquad x \to \pm\infty$$

$$\overline{\psi}^{(2)} = \frac{W_1^{(2)}(\{c_1, f_1\}, \{c_1, S_1 f_1\}; \psi)}{\Delta} \sim \exp[\mathrm{i}k(x + 4k^2 t + \widetilde{x})], \qquad x \to \pm\infty,$$

where $\theta = k(x + 4k^2 t + \widetilde{x})$, $\widetilde{x}$ is an analytic function of k.

Therefore, we have

$$\overline{\psi} = \begin{pmatrix} \overline{\psi}^{(1)} \\ \overline{\psi}^{(2)} \end{pmatrix} \sim \begin{pmatrix} \exp[-\mathrm{i}k(x + 4k^2 t + \widetilde{x})] \\ \exp[\mathrm{i}k(x + 4k^2 t + \widetilde{x})] \end{pmatrix}, \qquad x \to \pm\infty. \tag{4.9}$$

Furthermore, the asymptotic behavior of another independent solution $\widetilde{\psi}$ constructed by means of $\overline{\psi}$ reads

$$\widetilde{\psi} = \begin{pmatrix} \overline{\psi}^{(2)*} \\ -\overline{\psi}^{(1)*} \end{pmatrix} \sim \begin{pmatrix} \exp[-\mathrm{i}k(x + 4k^2 t + \widetilde{x})] \\ -\exp[\mathrm{i}k(x + 4k^2 t + \widetilde{x})] \end{pmatrix}, \qquad x \to \pm\infty. \tag{4.10}$$

We now define the Jost solution of the system (3.7) with $\lambda = \mathrm{i}k$ by imposing the asymptotic behavior

$$E_1 \sim \begin{pmatrix} 0 \\ 1 \end{pmatrix} \exp(\mathrm{i}kx), \qquad E_2 \sim \begin{pmatrix} 1 \\ 0 \end{pmatrix} \exp(-\mathrm{i}kx), \qquad x \to +\infty \tag{4.11a}$$

$$F_1 \sim \begin{pmatrix} 1 \\ 0 \end{pmatrix} \exp(-\mathrm{i}kx), \qquad F_2 \sim \begin{pmatrix} 0 \\ -1 \end{pmatrix} \exp(\mathrm{i}kx), \qquad x \to -\infty. \tag{4.11b}$$

The reflection coefficients $a(k), d(k)$ and the transmission coefficients $b(k), c(k)$ are defined by the formula

$$F_1 = a(k)E_1 + b(k)E_2, \qquad F_2 = c(k)E_1 + d(k)E_2. \tag{4.12}$$

In the one-positon potential case, we have

$$E_1 = \frac{1}{2B}\overline{\psi} - \frac{1}{2B}\widetilde{\psi}, \qquad E_2 = \frac{1}{2A}\overline{\psi} + \frac{1}{2A}\widetilde{\psi}, \tag{4.13}$$

where $A = \exp[-\mathrm{i}k(4k^2t + \widetilde{x})]$, $B = \exp[\mathrm{i}k(4k^2t + \widetilde{x})]$.

Therefore, according to the asymptotic behavior of (4.9), (4.10) and (4.11), we find from (4.12) and (4.13) that

$$a(k) = d(k) = 0, \qquad b(k) = 1, \qquad c(k) = -1. \tag{4.14}$$

Reflectionless potentials are characterized by the vanishing reflection coefficients $a(k), d(k)$ while the transmission coefficients $b(k)$, $c(k)$ are not unity for these solutions. Since $b(k) = 1$ and $c(k) = -1$ for a positon solution, the positon potentials are called the superreflectionless or supertransparent one.

4.3.2. Two-positon solution. The two-positon solution of (3.6) with $n = 2$, $\lambda_j = \mathrm{i}k_j$, $\mathrm{Im}\, k_j = 0$, $j = 1, 2$, is given by (3.9*c*)–(3.9*e*) with $q = 0$, $n = 0$, $N = 2$ and

$$f_j = \begin{pmatrix} \exp(-\mathrm{i}\theta_j) \\ \exp(\mathrm{i}\theta_j) \end{pmatrix} \theta_j = k_j\left(x + 4k_j^2 t + x_j\right), \qquad c_j(t) = a_j t + b_j, \qquad j = 1, 2$$

$$r_j = x + \tilde{x}_j + \left(12k_j^2 + a_j\right)t + b_j, \qquad \tilde{x}_j = x_j + k_j \partial_{k_j} x_j.$$

We easily obtain the asymptotic behavior of the two-positon solution for fixed θ_1 as $t \to \pm\infty$ (which implies $\theta_2 \to \pm\infty$)

$$q \sim -\frac{4k_1(\sin 2\theta_1 - 2k_1 r_1 \cos 2\theta_1)}{\sin^2(2\theta_1) - 4k_1^2 r_1^2}$$

$$\varphi_1[2] \sim \begin{pmatrix} \dfrac{2\sqrt{a_1}k_1(2k_1 r_1 \exp(-\mathrm{i}\theta_1) - \exp(\mathrm{i}\theta_1)\sin(2\theta_1))}{\sin^2(2\theta_1) - 4k_1^2 r_1^2} \\ \dfrac{2\sqrt{a_1}k_1(2k_1 r_1 \exp(\mathrm{i}\theta_1) - \exp(-\mathrm{i}\theta_1)\sin(2\theta_1))}{\sin^2(2\theta_1) - 4k_1^2 r_1^2} \end{pmatrix},$$

$$\varphi_2[2] \sim \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \qquad t \to \pm\infty.$$

When θ_2 is fixed and $t \to \pm\infty$ ($\theta_1 \to \pm\infty$), we have a similar result for the asymptotic behavior of the solution. Thus, we have shown that the two positons are totally insensitive to the mutual collision, even without additional phase shifts, which is intrinsic for the collision of two solitons.

4.3.3. Multipositon solutions. The N-positon solution of equation (3.6) with $n = N$ and $\lambda_j = ik_j$, $\mathrm{Im}\, k_j = 0$, $j = 1, \ldots, N$, is given by (3.9*c*) and (3.9*e*) with $q = 0$, $n = 0$ and

$$f_j = \begin{pmatrix} \exp(-\mathrm{i}\theta_j) \\ \exp(\mathrm{i}\theta_j) \end{pmatrix} \theta_j = k_j\left(x + 4k_j^2 t + x_j\right), \qquad c_j(t) = a_j t + b_j, \qquad j = 1, \ldots, N.$$

4.4. Multisoliton–positon, multisoliton–negaton and multipositon–negaton solutions

Like KdV equation, mKdV equation and KdVESCSs, the mKdV$^-$ESCSs has multisoliton–positon, multisoliton–negaton and multipositon–negaton solutions.

(1) The N-positon–M-soliton solution of equation (3.6) with $n = N+M$ and $\lambda_j = \mathrm{i}k_j$, $\mathrm{Im}\,k_j = 0$, $j = 1, \ldots, N$, real $\lambda_{N+m}, m = 1, \ldots, M$ are given by (3.9c)–(3.9e) with N replaced by $N+M$, $q = 0$, $n = 0$ and

$$f_j = \begin{pmatrix} \exp(-\mathrm{i}\theta_j) \\ \exp(\mathrm{i}\theta_j) \end{pmatrix}, \qquad \theta_j = k_j\left(x + 4k_j^2 t + x_j\right),$$
$$c_j(t) = a_j t + b_j, \qquad j = 1, \ldots, N$$
$$f_{N+m} = \begin{pmatrix} \exp(-\theta_{N+m}) \\ 0 \end{pmatrix}, \qquad \theta_{N+m} = \lambda_{N+m}\left(x - 4\lambda_{N+m}^2 t + x_{N+m}\right)$$
$$c_{N+m}(t) = -\frac{\exp(-\lambda_{N+m}\alpha_{N+m}t)}{4\lambda_{N+m}}, \qquad \alpha_{N+m} \geqslant 0, \qquad m = 1, \ldots, M.$$

(2) The N-negaton–M-soliton solution of equation (3.6), with $n = N + M$and real λ_j, $j = 1, \ldots, N+M$, is given by equations (3.9c) and (3.9e) with N replaced by $N+M$, $q = 0$, $n = 0$ and

$$f_j = \begin{pmatrix} \exp(-\theta_j) \\ \exp\theta_j \end{pmatrix}, \qquad \theta_j = \lambda_j\left(x - 4\lambda_j^2 t + x_j\right),$$
$$c_j(t) = a_j t + b_j, \qquad j = 1, \ldots, N$$
$$f_{N+m} = \begin{pmatrix} \exp(-\theta_{N+m}) \\ 0 \end{pmatrix}, \qquad \theta_{N+m} = \lambda_{N+m}\left(x - 4\lambda_{N+m}^2 t + x_{N+m}\right),$$
$$c_{N+m}(t) = -\frac{\exp(-\lambda_{N+m}\alpha_{N+m}t)}{4\lambda_{N+m}}, \qquad \alpha_{N+m} \geqslant 0, \qquad m = 1, \ldots, M.$$

(3) The N-positon–M-negaton solution of equation (3.6), with $n = N + M$ and $\lambda_j = \mathrm{i}k_j$, $\mathrm{Im}\,k_j = 0$, $j = 1, \ldots, N$ and real $\lambda_{N+m}, m = 1, \ldots, M$, is given by (3.9$c$)–(3.9$e$) with N replaced by $N+M$, $q = 0$, $n = 0$ and

$$f_j = \begin{pmatrix} \exp(-\mathrm{i}\theta_j) \\ \exp(\mathrm{i}\theta_j) \end{pmatrix}, \qquad \theta_j = k_j\left(x + 4k_j^2 t + x_j\right), \qquad j = 1, \ldots, N,$$
$$f_{N+m} = \begin{pmatrix} \exp(-\theta_{N+m}) \\ \exp\theta_{N+m} \end{pmatrix}, \qquad \theta_{N+m} = \lambda_{N+m}\left(x - 4\lambda_{N+m}^2 t + x_{N+m}\right), \qquad m = 1, \ldots, M$$
$$c_j(t) = a_j t + b_j, \qquad j = 1, \ldots, N+M,$$

where $a_j \neq 0$ and b_j are real constants.

We can analyze the interaction of the soliton and the positon, the soliton and the negaton, the positon and the negaton in a similar way as in [31, 32]. We would like to point out that the results of the analysis will be almost the same as in [31, 32], so we omit it.

5. Conclusion

In this paper, by reducing the generalized binary Darboux transformation with an arbitrary function at time t for the AKNSECSs presented in [20], we obtain the generalized binary Darboux transformation with arbitrary functions at time t for the mKdV$^-$ESCSs which offers a non-auto-Bäcklund transformation between two mKdV$^-$ESCSs with different degrees of sources and enables us to construct some general solutions with Narbitrary t-functions for the mKdV$^-$ESCSs. Some new solutions for the mKdV$^-$ESCSs such as singular multisoliton, multipositon, multinegaton, multisoliton–positon, multisoliton–negaton and multipositon–negaton solutions are constructed by taking the special initial solution for auxiliary linear problems and the special functions of time t. In addition, the properties of these solutions are analyzed in detail.

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References

[1] Leon J and Latifi A 1990 *J. Phys. A: Math Gen.* **23** 1385
[2] Mel'nikov V K 1992 *Inverse Problems* **8** 133
[3] Doktorov E V and Vlasov R A 1983 *Opt. Acta* **30** 223
[4] Claude C, Latifi A and Leon J 1991 *J. Math. Phys.* **32** 3321
[5] Mel'nikov V K 1989 *Commun. Math. Phys.* **126** 201
[6] Mel'nikov V K 1990 *Inverse Problems* **6** 233
[7] Kaup D J 1987 *Phys. Rev. Lett.* **59** 2063
[8] Doktorov E V and Shchesnovich V S 1995 *Phys. Lett.* A **207** 153
[9] Shchesnovich V S and Doktorov E V 1996 *Phys. Lett.* A **213** 23
[10] Mel'nikov V K 1990 *J. Math. Phys.* **31** 1106
[11] Leon J 1990 *Phys. Lett.* A **144** 444
[12] Zeng Y B, Ma W X and Lin R 2000 *J. Math. Phys.* **41** 5453
[13] Zeng Y B 1995 *Acta Math. Sin.* **15** 337
[14] Zeng Y B and Li Y S 1993 *J. Phys A: Math. Gen.* **26** 273
[15] Zeng Y B 1994 *Physica* D **73** 171
[16] Zeng Y B and Li Y S 1996 *Acta Math. Sin. New Ser.* **12** 217
[17] Zeng Y B, Ma W X and Shao Y J 2001 *J. Math. Phys.* **42** 2113
[18] Zeng Y B, Shao Y J and Xue W M 2003 *J. Phys. A: Math. Gen.* **36** 5035
[19] Xiao T and Zeng Y B 2004 *J. Phys. A: Math.Gen.* **37** 7143
[20] Shao Y J and Zeng Y B 2005 *J. Phys. A: Math. Gen.* **38** 2441
[21] Liu X J and Zeng Y B 2005 *J. Phys. A: Math. Gen.* **38** 8951
[22] Zeng Y B, Shao Y J and Ma W X 2002 *Commun. Theor. Phys.* **38** 641
[23] Zhang D J 2003 *Chaos Solitons Fractals* **18** 31
[24] Zhang D J 2002 *J. Phys. Soc. Japan* **71** 2649
[25] Zhang D J 2003 *Physica* A **321** 467
[26] Hu X B 1996 *Chaos Solitons Fractrals* **7** 211
[27] Hu X B and Wang H Y 2006 *Inverse Problems* **22** 1903C1920
[28] Matveev V B and Salle M A 1991 *Darboux Transformations and Solitons* (Springer: Berlin)
[29] Matveev V B 2002 *Theor. Math. Phys.* **131** 483
[30] Ma W X 2004 *Chaos Solitons Fractals* **19** 163
[31] Raisinariu C, Sukhutame U and Khare K 1996 *J. Phys. A: Math. Gen.* **29** 1803
[32] Stahlhofen A 1992 *Ann. Phys.* **1** 554
[33] Barran S, Kovlayov M and Khare A 1999 *J. Phys. A: Math. Gen.* **32** 6121
[34] Beutler R 1993 *J. Math. Phys.* **34** 3098
[35] Ma W X 2005 *Chaos Solitons Fractals* **26** 1453
[36] Ma W X 2003 *J. Phys. Soc. Japan* **72** 3017
[37] Ma W X and You Y C 2005 *Trans. Am. Math. Soc.* **357** 1753
[38] Ma W X 2002 *Phys. Lett.* A **301** 35
[39] Ma W X and Maruno K 2004 *Physica* A **343** 219